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FUNDAMENTAL GROUPS OF MANIFOLDS OF NONPOSITIVE CURVATURE

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Introduction

The universal covering space H of a complete Riemannian manifold M of nonpositive sectional curvature is diffeomorphic to $\mathbf{R}^n$, $n = \dim(M)$. Hence the homotopy type of M is completely determined by the isomorphism class of the fundamental group Γ of M. It is, therefore, only natural to expect strong relations between the geometric structure of M and the algebraic structure of Γ. In this paper we obtain several such relations:

A general assumption in the results we state below is that

(1) the sectional curvature is nonpositive and bounded from below by some constant $-a^2$ and

(2) the volume of M is finite.

We define the rank of a unit tangent vector v of M, rank(v), to be the dimension of the space of all parallel Jacobi fields along the geodesic γ_v which has initial velocity v. The minimum of rank(v) over all $v \in SM$ is called the rank of M. This agrees with the usual rank if M is a locally symmetric space. Manifolds of rank one resemble manifolds of strictly negative curvature (see [2] [3], and §2 below). Manifolds of higher rank are studied in [5], [6], [7], and [3], and the conclusive result is that H is a space of rank one, or a symmetric space, or a Riemannian product of such spaces. This is the basic ingredient in the proofs of our results; it allows us, more or less, to consider only the cases that H is of rank one or a symmetric space.

As a first example of this principle we indicate in our preliminary section the proof of the following theorem.

Theorem A. *Either M is flat or Γ contains a nonabelian free subgroup.*

This is an improvement of the result of Avez [1] that Γ has exponential growth if M is compact and not flat.

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Following Prasad and Raghunathan [27] we define the subset A_i of Γ to consist of those elements whose centralizer contains a free abelian group of rank at most i as a subgroup of finite index. Note that

$$A_0 \subset A_1 \subset A_2 \subset \cdots .$$

We define $r(\Gamma)$ to be the minimal integer i such that Γ can be written as a finite union

$$\Gamma = \gamma_1 A_i \cup \gamma_2 A_i \cup \cdots \cup \gamma_m A_i$$

of left translates of $A_i, \gamma_1, \cdots, \gamma_m \in \Gamma$ arbitrary.

A result of Prasad and Raghunathan in [27] states that $r(\Gamma)$ is equal to the rank of the symmetric space G/K of noncompact type if Γ belongs to the identity component of G. (This does not imply that $r(\Gamma) = \operatorname{rank}(M)$ if M is a locally symmetric space, cf. §4.) We set

$$\operatorname{rank}(\Gamma) = \max\{r(\Gamma^*) \mid \Gamma^* \text{ a finite index subgroup of } \Gamma\}.$$

Theorem B. $\operatorname{rank}(M) = \operatorname{rank}(\Gamma)$.

Since the homotopy type of M is determined by the fundamental group Γ we obtain the following immediate consequences of Theorem B.

Corollary 1. *The rank of M is a homotopy invariant of M.*

Corollary 2. *If M is compact, then the ergodicity or nonergodicity of the geodesic flow in the unit tangent bundle SM is a homotopy invariant of M.*

In the special case that M is compact and of rank one the first corollary was obtained in [10]. With regard to the second corollary it was shown in [4] that the geodesic flow is ergodic if M is compact and of rank one, and the converse assertion was proved in Theorem 4.5 of [6].

As a further application of Theorem B we obtain a characterization of irreducible locally symmetric spaces of noncompact type and rank at least two in terms of algebraic data in the fundamental group. Here a Riemannian manifold is called irreducible if none of its finite coverings is a Riemannian product.

Theorem C. *M is an irreducible locally symmetric space of noncompact type of rank $k \geqslant 2$ if and only if Γ satisfies the following three conditions*:

(1) *Γ is finitely generated.*

(2) *No finite index subgroup of Γ is a direct product.*

(3) $\operatorname{rank}(\Gamma) = k \geqslant 2$.

Remark. Clearly condition (1) can be deleted if M is assumed to be compact. In combination with (2) it implies that the universal cover H does not contain a Euclidean factor (see (1.9) below). One may replace (1) by the assumption that Γ does not contain a proper normal abelian subgroup; this also excludes Euclidean factors in H by the main theorem of [13].

Applying the celebrated rigidity results of Mostow [26] and Margulis [24], [25] Theorem C has the following consequence.

Theorem D. *Suppose Γ is isomorphic to the fundamental group Γ^* of an irreducible locally symmetric space M^* of noncompact type and rank at least two. Then M and M^* are isometric up to rescalings of the metric of M^*.*

Under the assumption that M is compact this was proved by Gromov [18], [19], and, in a special case, by Eberlein [14].

The paper is organized as follows. §1 introduces notation and relevant results, and we use these results to prove Theorem A. In §2 we define and discuss the rank of an abstract group. Our main result here is that the rank of a direct product is the sum of the ranks of the factors, and that the rank is unchanged under passage to a finite index subgroup. Combined with (1.7) and (1.8) below this reduces Theorem B to the two cases that M has rank one or is locally symmetric, the second of which is Theorem 3.9 of [27]. In §3 we prove Theorems B and C. §4 contains examples. The first two examples illustrate the fact that Theorem B is false if rank(Γ) is replaced by $r(\Gamma)$, even if $M = H/\Gamma$ is locally symmetric. The third example shows the necessity of the first hypothesis of Theorem C.

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1. Preliminaries

(1.1) Notation. In general we shall use the notation and basic results of [15]. See also §1 of [5]. In this paper all manifolds considered are complete, connected Riemannian manifolds of nonpositive sectional curvature. M will always denote a manifold with finite volume and H will always denote a simply connected manifold. The unit tangent bundles are denoted by SM, SH, and $\pi: SM \to M$, $SH \to H$ denotes the projection map. All geodesics of M or H are assumed to have unit speed. If v is a unit vector in SM or SH, then γ_v denotes the geodesic with initial velocity v. The Riemannian metric on M, H induces a natural Riemannian metric on SM, SH, and the corresponding distance functions are denoted by d on M, H and by d^* on SM, SH. The isometry group of H is denoted by $I(H)$, and the connected component of $I(H)$ that contains the identity is denoted by $I_0(H)$.

(1.2) Asymptotes and points at infinity. Two unit speed geodesics γ, σ of H are said to be *asymptotes* if $d(\gamma t, \sigma t) \leqslant c$ for some positive constant c and for all $t \geqslant 0$. An equivalence class of asymptotes is a *point at infinity* for H, and $H(\infty)$ denotes the set of all points at infinity for H. We let $\gamma(\infty)$ and $\gamma(-\infty)$ denote the points at infinity determined by γ and $\gamma^{-1}: t \to \gamma(-t)$. Given points

$p \in H$ and $x \in H(\infty)$ there is a unique unit speed geodesic γ_{px} such that $\gamma_{px}(0) = p$ and γ_{px} belongs to x. The space $\overline{H} = H \cup H(\infty)$ with a natural topology is homeomorphic to the closed unit n-ball, and for each point p in H the map $x \to \gamma'_{px}(0)$ is a homeomorphism of $H(\infty)$ onto the unit $(n-1)$-sphere in T_pH. Isometries of H extend in an obvious way to homeomorphisms of $H(\infty)$.

Given $p \in H$ and $x \in H(\infty)$ the restriction to $[0, \infty]$ of γ_{px} is said to join p and x or to have endpoints p and x. Similarly, if x, y are distinct points in $H(\infty)$, then a maximal geodesic γ is said to join x and y or to have endpoints x and y if $\gamma(\infty) = x$ and $\gamma(-\infty) = y$ (or $\gamma(\infty) = y$ and $\gamma(-\infty) = x$). The geodesic (when this is unique) joining distinct points x, y in $\overline{H}$ will be denoted by γ_{xy}. Our convention is that $\gamma_{xy}(0) = x$ if x is a point of H.

(1.3) Rank of a manifold [5]. For each unit vector v in SH or SM we define $r(v)$ to be the dimension of the space of parallel Jacobi vector fields defined on the maximal geodesic $\gamma_v: \mathbf{R} \to H$ or $\mathbf{R} \to M$. We define rank(H) (or rank(M)) to be the minimum of the integers $r(v)$ over all vectors v in SH (or SM). Clearly rank(H) = rank(M) if H covers M, and the rank of a product manifold is the sum of the ranks of the factors. A vector v in SH or SM is called *regular* if $r(v) =$ rank(H) or rank(M). If H admits a smooth quotient manifold M of finite volume, then the regular vectors of SH (SM) form a dense open subset of SH (SM) by Theorem 2.6 of [5].

(1.4) The De Rham decomposition [22]. A complete, simply connected Riemannian manifold X is irreducible if it cannot be written as a Riemannian product $X_1 \times X_2$ of two Riemannian manifolds of positive dimension. A reducible space X has a Riemannian product decomposition

$$X = X_0 \times X_1 \times \cdots \times X_k,$$

where X_0 is a Euclidean space and each X_i is irreducible for $1 \leqslant i \leqslant k$. This decomposition of de Rham is unique up to isometric equivalence and ordering of the factors $X_1, \cdots, X_k$.

If $H = H_1 \times \cdots \times H_k$ is any Riemannian product decomposition of a simply connected manifold H of nonpositive curvature, then a subgroup $\Gamma \subseteq I(H)$ is said to preserve the factors of the decomposition if the foliations of H induced by the factors are left invariant by Γ. In this case every element φ of Γ can be written $\varphi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_k$, where $\varphi_i \in I(H_i)$, and one obtains projection homomorphisms $p_i: \Gamma \to I(H_i)$ given by $p_i(\varphi) = \varphi_i$ for all $1 \leqslant i \leqslant k$. If H has no Euclidean de Rham factor, then each factor H_i above is a Riemannian product of some subset of the de Rham factors of H. Since every subgroup Γ of $I(H)$ has a finite index subgroup Γ^* that preserves the factors of the de Rham decomposition of H, it follows that Γ^* preserves the factors of all product decompositions of H whenever H has no Euclidean de Rham factor.

(1.5) Duality condition [8], [2]. Two points x, y in $H(\infty)$ are said to be *dual* relative to a group $\Gamma \subseteq I(H)$ if there exists a sequence $\{\varphi_n\} \subseteq \Gamma$ such that $\varphi_n(p) \to x$ and $\varphi_n^{-1}(p) \to y$ as $n \to \infty$ for any point p in H. Given a point $x \in H(\infty)$ the set of points y in $H(\infty)$ that are dual to x relative to Γ is closed in $H(\infty)$ and invariant under Γ. A group $\Gamma \subseteq I(H)$ is said to satisfy the *duality condition* if for every geodesic γ of H the points $\gamma(\infty)$ and $\gamma(-\infty)$ are dual relative to Γ.

The duality condition may be restated in the following useful form (cf. [2, p. 137]). For any group $\Gamma \subseteq I(H)$ one defines a nonwandering set $\Omega(\Gamma) \subseteq SH$ as follows: A vector $v \in SH$ lies in $\Omega(\Gamma)$ if and only if for every neighborhood $O \subseteq SH$ of v and every positive number A there exists $t \geqslant A$ and $\varphi \in \Gamma$ such that $[(d\varphi \circ g^t)(0)] \cap O$ is nonempty, where $\{g^t\}$ denotes the geodesic flow. The condition that Γ satisfy the duality condition is then precisely the condition that $\Omega(\Gamma) = SH$.

If $\Gamma \subseteq I(H)$ is a discrete group such that the quotient space H/Γ has finite volume, then Γ satisfies the duality condition. For any group Γ that satisfies the duality condition one also has the following useful properties:

(1) If Γ^* is a finite index subgroup of a group Γ that satisfies the duality condition, then Γ^* satisfies the duality condition.

(2) If H is a Riemannian product $H_1 \times H_2$ and if $\Gamma \subseteq I(H_1 \times H_2)$ is a subgroup that satisfies the duality condition and preserve the factors of the decomposition, then $p_i(\Gamma)$ is a subgroup of $I(H_i)$ that satisfies the duality condition for $i = 1, 2$, where $p_i: \Gamma \to I(H_i)$ is the projection homomorphism.

The projection $p_i(\Gamma)$ need not be discrete in $I(H_i)$ even if Γ is discrete in $I(H)$. The restriction in (2) that Γ preserve the factors of the decomposition is usually not serious. One can usually reduce a given problem to the case that H has no Euclidean factor (cf. the proof of Theorem B in §3). By (1.4) one may then pass to a subgroup Γ^* of finite index which preserves the factors of the decomposition, and by (1) above Γ^* still satisfies the duality condition.

Splitting theorems. If $\Gamma \subseteq I(H)$ satisfies the duality condition, then an algebraic splitting of Γ corresponds very closely to a geometric splitting of H.

(1.6) Theorem. *Let H admit no Euclidean de Rham factor, and let $\Gamma \subseteq I(H)$ satisfy the duality condition. If Γ is a direct product $\Gamma_1 \times \Gamma_2$, then H splits as a Riemannian product $H_1 \times H_2$ such that $\Gamma_1 \subseteq I(H_1) \times \{1\}$ and $\Gamma_2 \subseteq \{1\} \times I(H_2)$. In particular, if Γ is discrete and H/Γ is a smooth manifold of finite volume, then H/Γ is isometric to $(H_1/\Gamma_1) \times (H_2/\Gamma_2)$.*

(1.7) Theorem. *Let H be a Riemannian product $H_1 \times H_2$ such that H has no Euclidean de Rham factor and $I(H_2)$ is discrete. Let $\Gamma \subseteq I(H)$ be a discrete* *group such that the quotient space H/Γ has finite volume. Then Γ has a finite index subgroup Γ^* that is a direct product $\Gamma_1^* \times \Gamma_2^*$, where $\Gamma_1^* \subseteq I(H_1) \times \{1\}$ and $\Gamma_2^* \subseteq \{1\} \times I(H_2)$. In particular if H/Γ is a smooth manifold, then H/Γ^* is a Riemannian product $(H_1/\Gamma_1^*) \times (H_2/\Gamma_2^*)$ that finitely covers H/Γ.*

The first of these results is a special case of the corollary to Theorem 1 of [29]. The result in [29] itself generalizes results of [16] and [23], who proved the result in the case that Γ is a discrete centerless group and H/Γ is a smooth compact manifold. The second of the results stated above follows from Theorem 4.1 and Proposition 2.2 of [9]. These two results of [9] are actually stated for discrete groups Γ such that the quotient space H/Γ is a smooth manifold of finite volume, but the proofs remain valid in this generality and require at most trivial modifications.

(1.8) Theorem. *Let H have sectional curvature satisfying $-a^2 \leqslant K \leqslant 0$ for some positive constant a, and let $\Gamma \subseteq I(H)$ be a discrete group such that the quotient space H/Γ has finite volume. Then H is a Riemannian product*

$$H = H_0 \times H_S \times H_1 \times \cdots \times H_k,$$

where H_0 is a Euclidean space with its canonical flat metric, H_S is a symmetric space of noncompact type, and, for $1 \leqslant i \leqslant k$, H_i is a rank one manifold such that $I(H_i)$ is discrete and satisfies the duality condition.

Of course, any of the factors in the decomposition above could be absent. The result may be true without the bounded curvature restriction and with only the single hypothesis that $I(H)$ satisfy the duality condition. The result as stated follows immediately from Proposition 4.1 of [12] and the discussion in §1 of [3], especially Theorem 1.4 (see also [7]).

Spaces with Euclidean de Rham factors.

(1.9) Theorem. *Let H be a Riemannian product $E^k \times H_1$, where E^k is a flat Euclidean space of dimension $k \geqslant 1$ and H_1 has no Euclidean de Rham factor. Let $\Gamma \subseteq I(H)$ be a finitely generated discrete group such that $M = H/\Gamma$ is a smooth manifold of finite volume. Then M admits a finite Riemannian cover that is diffeomorphic (but not necessarily isometric) to a Riemannian product $T^k \times M_1$, where T^k denotes a flat k-torus and M_1 is a smooth Riemannian quotient of H_1.*

The special case in which H/Γ is compact is Corollary 2 of [13]. The proof of that result actually proves the result stated above. The hypothesis that Γ be finitely generated cannot be deleted (see §4).

The results (1.6), (1.7), (1.8), and (1.9) will be needed later to prove Theorems B and C of the Introduction. We now use the third of these results to prove Theorem A. We assume that $M = H/\Gamma$ is not flat.

Proof of Theorem A. By Theorem 1.8 and the hypotheses of Theorem A we may write H as a Riemannian product

$$H = H_0 \times H_S \times H_1 \times \cdots \times H_k$$

as described in Theorem 1.8, where either the symmetric space factor H_S exists or the rank one factor H_1 exists. Let Γ^* be a finite index subgroup of Γ that preserves the factors of the decomposition, and let $p_0 : \Gamma^* \to I(H_0)$, $p_S : \Gamma^* \to I(H_S)$, and $p_i : \Gamma^* \to I(H_i)$ for $1 \leqslant i \leqslant k$ be the corresponding projection homomorphisms.

If H admits a rank one de Rham factor H_1, then the group $p_1(\Gamma^*)$ satisfies the duality condition in H_1 (see (1.5)), and hence $p_1(\Gamma^*)$ contains a nonabelian free subgroup F_1 by Theorem 3.2 of [2]. It then follows that $F = p_1^{-1}(F_1)$ contains a nonabelian free subgroup in Γ^*.

If H admits a symmetric space factor H_s, then we consider the group $\Gamma^{**} = p_S^{-1}[p_S(\Gamma^*) \cap I_0(H_S)]$. Note that Γ^{**} has finite index in Γ^* since $I_0(H_S)$ has finite index in $I(H_S)$, and clearly $p_S(\Gamma^{**}) \subseteq I_0(H_S)$. The group $I_0(H_S)$ is a semisimple Lie group with trivial center and no compact factors, and hence it may be identified faithfully under the adjoint representation with a subgroup of $\mathrm{SL}(n, \mathbf{R})$, $n = \dim I_0(H)$. By Theorem 1 of [30] (see also [20]) $p_s(\Gamma^{**})$ either contains a nonabelian free subgroup F_S or admits a solvable subgroup of finite index. It suffices to prove that the second possibility cannot occur, for then $F = p_S^{-1}(F_S)$ will contain a nonabelian free subgroup of Γ^{**}. The group $p_S(\Gamma^{**})$ satisfies the duality condition in H_S by earlier remarks in (1.5) since Γ^{**} has finite index in Γ. If $p_S(\Gamma^{**})$ did admit a solvable subgroup of finite index, then H_S would be isometric to a Euclidean space by Theorem 5.1 of [8], which is not the case.

2. The rank of a group

Let Γ be an abstract group. Given an element φ in Γ, let $Z_\Gamma(\varphi)$ denote the centralizer of φ in Γ. As in the Introduction we denote by $A_i(\Gamma)$ the subset of Γ that consists of elements φ such that $Z_\Gamma(\varphi)$ contains a free abelian subgroup of rank $\leqslant i$ as a subgroup of finite index. We define

$$r(\Gamma) = \min\Big\{ i : \text{there exist finitely many elements } \varphi_j \in \Gamma \text{ such that } \Gamma = \bigcup_j \varphi_j \cdot A_i(\Gamma) \Big\}$$

and set

$$\operatorname{rank}(\Gamma) = \sup\{ r(\Gamma^*) : \Gamma^* \subseteq \Gamma \text{ is a finite index subgroup} \}.$$

We allow the possibility that $r(\Gamma) = \operatorname{rank}(\Gamma) = 0$ (e.g. Γ is finite), and we set $r(\Gamma) = \operatorname{rank}(\Gamma) = +\infty$ if the sets $A_i(\Gamma)$ are empty or if Γ is not covered by finitely many translates of any of the sets $A_i(\Gamma)$. We are primarily interested in the case that Γ is a countably infinite group that arises as a discrete group of isometries of some nonpositively curved space H. The distinction between $r(\Gamma)$ and $\operatorname{rank}(\Gamma)$ is a necessary technical complication (see §4).

The main result of this section is the following:

(2.1) Proposition. *Let Γ be an abstract group.*

(1) *If Γ^* is a finite index subgroup of Γ, then* $\operatorname{rank}(\Gamma^*) = \operatorname{rank}(\Gamma)$.

(2) *If $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$, a direct product, then*

$$r(\Gamma) = \sum_{i=1}^{k} r(\Gamma_i) \quad \textit{and} \quad \operatorname{rank}(\Gamma) = \sum_{i=1}^{k} \operatorname{rank}(\Gamma_i).$$

The proof of this result will be carried out in several steps.

(2.2) If $\Gamma^* \subseteq \Gamma$ is a finite index subgroup, then $r(\Gamma) \leqslant r(\Gamma^*)$ and $\operatorname{rank}(\Gamma) = \operatorname{rank}(\Gamma^*)$.

Proof. The inequality follows since $A_i(\Gamma^*) \subseteq A_i(\Gamma)$ for all i, and as a consequence we get $r(\Gamma) \leqslant r(\Gamma^*)$. As for the second claim, clearly $\operatorname{rank}(\Gamma) \geqslant \operatorname{rank}(\Gamma^*)$. If $\tilde{\Gamma}$ is a finite index subgroup of Γ, then $\tilde{\Gamma}^* = \Gamma^* \cap \tilde{\Gamma}$ is a finite index subgroup of Γ^* and of $\tilde{\Gamma}$. Hence $\operatorname{rank}(\Gamma^*) \geqslant r(\tilde{\Gamma}^*) \geqslant r(\tilde{\Gamma})$ and therefore $\operatorname{rank}(\Gamma^*) = \operatorname{rank}(\Gamma)$.

(2.3) $r(\Gamma_1 \times \Gamma_2) \leqslant r(\Gamma_1) + r(\Gamma_2)$.

This follows from the fact that

$$A_i(\Gamma_1) \times A_j(\Gamma_2) \subseteq A_{i+j}(\Gamma_1 \times \Gamma_2).$$

(2.4) Let $\Gamma = \Gamma_1 \times \Gamma_2$ and let $\varphi = \varphi \times \varphi_2 \in A_i(\Gamma)$. Then there exist integers r and s such that $r + s = i$, $\varphi_1 \in A_r(\Gamma_1)$, and $\varphi_2 \in A_s(\Gamma_2)$.

Proof. By hypothesis $Z_\Gamma(\varphi) = Z_{\Gamma_1}(\varphi_1) \times Z_{\Gamma_2}(\varphi_2)$ contains a free abelian subgroup Δ of rank r^* as a subgroup of finite index for some $r^* \leqslant i$. For $j = 1, 2$ choose free abelian subgroups $\Delta_j \subseteq Z_{\Gamma_j}(\varphi_j)$ of rank r_j such that $\Delta_1 \times \Delta_2$ has finite index in Δ and Δ_j has finite index in $Z_{\Gamma_j}(\varphi_j)$. It follows that $r_1 + r_2 = r^*$ since $\Delta_1 \times \Delta_2$ has finite index in Δ. Now choose integers $r \geqslant r_1$ and $s \geqslant r_2$ such that $r + s = i$.

As a consequence of (2.4) and the remark following (2.3) we obtain

$$\begin{aligned} A_i(\Gamma_1 \times \Gamma_2) &= \bigcup_{r+s=i} A_r(\Gamma_1) \times A_s(\Gamma_2) \\ &\subseteq \{A_i(\Gamma_1) \times \Gamma_2\} \cap \{\Gamma_1 \times A_i(\Gamma_2)\}. \end{aligned} \tag{2.5}$$

Next we show that

$$r(\Gamma_1 \times \Gamma_2) \geqslant r(\Gamma_k) \quad \text{for } k = 1, 2. \tag{2.6}$$

To prove this merely note that if $\Gamma_1 \times \Gamma_2 = \bigcup_j (\varphi_j \times \psi_j) \cdot A_i(\Gamma_1 \times \Gamma_2)$ for some finite collection $\{\varphi_j \times \psi_j\}$ of elements of $\Gamma_1 \times \Gamma_2$, then $\Gamma_1 = \bigcup_j \varphi_j \cdot A_i(\Gamma_1)$ and $\Gamma_2 = \bigcup_j \psi_j \cdot A_i(\Gamma_2)$ by (2.5).

(2.7) $$r(\Gamma_1 \times \Gamma_2) = r(\Gamma_1) + r(\Gamma_2).$$

Proof. By (2.3) and (2.6) it suffices to consider the case that the integers $r(\Gamma_1 \times \Gamma_2)$, $r(\Gamma_1)$, and $r(\Gamma_2)$ are all finite and positive. Let $\Gamma = \Gamma_1 \times \Gamma_2$ and suppose that

$$r(\Gamma) = i < r(\Gamma_1) + r(\Gamma_2).$$

Let $j_1 = r(\Gamma_1) - 1 \geqslant 0$ and $j_2 = r(\Gamma_2) - 1 \geqslant 0$. Then

(a) $$A_i(\Gamma) = \bigcup_{r+s=i} A_r(\Gamma_1) \times A_s(\Gamma_2) \subseteq \{A_{j_1}(\Gamma_1) \times \Gamma_1\} \cup \{\Gamma_1 \times A_{j_2}(\Gamma_2)\}.$$

Choose elements $\varphi_1 \times \psi_1, \cdots, \varphi_N \in \psi_N$ such that

(b) $$(\varphi_1 \times \psi_1) \cdot A_i(\Gamma) \cup \cdots \cup (\varphi_N \times \psi_N) \cdot A_i(\Gamma) = \Gamma.$$

By definition of j_1 and j_2 there are elements $\varphi_0 \in \Gamma_1$ and $\psi_0 \in \Gamma_2$ such that

$$\varphi_0 \notin \varphi_1 \cdot A_{j_1}(\Gamma_1) \cup \cdots \cup \varphi_N \cdot A_{j_1}(\Gamma_1),$$
$$\psi_0 \notin \psi_1 \cdot A_{j_2}(\Gamma_2) \cup \cdots \cup \psi_N \cdot A_{j_2}(\Gamma_2).$$

Then

$$\varphi_0 \times \psi_0 \notin (\varphi_1 \times \psi_1) \cdot (A_{j_1}(\Gamma_1) \times \Gamma_2) \cup \cdots \cup (\varphi_N \times \psi_N) \cdot (A_{j_1}(\Gamma_1) \times \Gamma_2)$$
$$\cup (\varphi_1 \times \psi_1) \cdot (\Gamma_1 \times A_{j_2}(\Gamma_2)) \cup \cdots \cup (\varphi_N \times \psi_N) \cdot (\Gamma_1 \times A_{j_2}(\Gamma_2)).$$

This contradicts (a) and (b).

Proof of Proposition 2.1. Assertion (1) is contained in (2.2), and the first part of assertion (2) is contained in (2.7). If Γ_i^* has finite index in Γ_i for $1 \leqslant i \leqslant k$, then $\Gamma_1^* \times \Gamma_2^* \times \cdots \times \Gamma_k^*$ has finite index in $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$. Hence

$$\operatorname{rank}(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k) \geqslant \sum_{i=1}^{k} r(\Gamma_i^*)$$

by (2.7), which proves that

$$\operatorname{rank}(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k) \geqslant \sum_{i=1}^{k} \operatorname{rank}(\Gamma_i).$$

Conversely, if Γ^* has finite index in Γ, then Γ^* contains a finite index subgroup of the form $\Gamma_1^* \times \cdots \times \Gamma_k^*$, where each Γ_i^* has finite index in Γ_i. Hence $r(\Gamma^*) \leqslant \sum_{i=1}^{k} \operatorname{rank}(\Gamma_i)$ by (2.2) and (2.7), which proves that

$$\operatorname{rank}(\Gamma) = \operatorname{rank}(\Gamma_1 \times \cdots \times \Gamma_k) \leqslant \sum_{i=1}^{k} \operatorname{rank}(\Gamma_i).$$

3. The main results

In the first part of this section we consider rank one manifolds. We say that a geodesic γ in M or in H has rank one if $\gamma'(t)$ has rank one for some t (and hence for all t). For a subgroup $\Gamma \subseteq I(H)$ define $B_1 = B_1(\Gamma)$ by

$$B_1 = \{\varphi \in \Gamma : \varphi \text{ translates a rank one geodesic}\}.$$

If $\varphi \in B_1$ translates a rank one geodesic γ, then γ is the unique geodesic translated by φ and hence is left invariant by $Z_\Gamma(\varphi)$. In particular, if Γ is discrete, then $Z_\Gamma(\varphi)$ contains an infinite cyclic subgroup of finite index, which proves that $B_1(\Gamma) \subseteq A_1(\Gamma)$.

(3.1) Theorem. *Let $\Gamma \subseteq I(H)$ be a discrete subgroup that satisfies the duality condition. If the (geometric) rank of H is one, then $r(\Gamma) = 1$.*

Remark. The proof actually shows that if Γ is any subgroup of $I(H)$ (possibly not discrete) that satisfies the duality condition, then there exist four elements $\varphi_1, \varphi_2, \psi_1$, and ψ_2 in Γ such that

$$\Gamma = \varphi_1 B_1 \cup \varphi_2 B_1 \cup \psi_1 B_1 \cup \psi_2 B_1.$$

The proof of the theorem will require some preliminary results. The following definition will be useful.

(3.2) Definition. A point $x \in H(\infty)$ is called a *hyperbolic point* if for any $y \neq x$, $y \in H(\infty)$, there is a rank one geodesic γ joining x and y (that is, $\gamma(\infty) = x$ and $\gamma(-\infty) = y$ or vice versa).

(3.3) Remark. If a rank one geodesic γ is translated by an isometry $\varphi \in I(H)$, then $\gamma(\infty)$ and $\gamma(-\infty)$ are hyperbolic points. This follows from a trivial modification of the proof of part (iii) of Theorem 2.2 in [2], where a slightly different result is proved. If H has rank one and $I(H)$ satisfies the duality condition, then the hyperbolic points of $H(\infty)$ that arise in this fashion are dense in $H(\infty)$. To see this one can either apply Theorem 2.13 of [2] and the fact that the rank one vectors are dense in SH or Theorem 2.13 of [2] and Lemma 2.3b of [9].

(3.4) Lemma. *Let γ be a rank one geodesic of H and choose $\varepsilon_0 > 0$ so that if $d^*(v, \gamma'(0)) < \varepsilon_0$, then γ_v is of rank one. Then for each ε with $0 < \varepsilon < \varepsilon_0$ there exist neighborhoods $V \subseteq \overline{H} = H \cup H(\infty)$ of $x = \gamma(\infty)$ and $U \subseteq \overline{H}$ of $y = \gamma(-\infty)$ such that for any points $x^* \in V$ and $y^* \in U$ there is a unique rank one geodesic γ^* in H joining x^* to y^* such that $d^*(\gamma^{*\prime}(0), \gamma'(0)) \leqslant \varepsilon$ for a suitable parametrization of γ^*.*

Proof. We need to show only the existence of γ^*. Rank one follows from the choice of ε_0 and uniqueness of γ^* follows from rank one.

Let $\{x_n^*\}$, $\{y_n^*\}$ be arbitrary sequences in H such that $x_n^* \to x$ and $y_n^* \to y$ as $n \to +\infty$. Let γ_n^* be the geodesic from x_n^* to y_n^*, parametrized so that $d(p, \gamma_n^*) = d(p, \gamma_n^*(0))$, where $p = \gamma(0)$. Since γ has rank one it follows from assertion $(*)$ in the proof of Lemma 2.1 in [2] that $d(p, \gamma_n^*) \to 0$, and hence $\gamma_n^{*\prime}(0) \to \gamma'(0)$ as $n \to +\infty$. By properties of the cone topology of $\overline{H}$ (§2 of [15]) the proof is complete.

(3.5) Lemma. *Let $p \in H$ be fixed and let $x \in H(\infty)$ be a hyperbolic point. For any neighborhood O^* of x in $\overline{H}$ there exists a neighborhood O of x in $\overline{H}$ and a number $R > 0$ such that if σ is a geodesic in H with endpoints in O and $\overline{H} - O^*$, then $d(p, \sigma) \leqslant R$.*

Proof. The set $\overline{H} - O^*$ is compact in $\overline{H}$, and there exists a geodesic from x to q for every $q \in \overline{H} - \{x\}$ since x is a hyperbolic point. Given a point $q \in \overline{H} - O^*$ we can find neighborhoods $O_q \subseteq \overline{H}$ of q and $V_q \subseteq \overline{H}$ of x and a number $R_q > 0$ such that if $x^* \in V_q$ and $q' \in O_q$, then there exists a geodesic σ from q' to x^* with $d(p, \sigma) \leqslant R_q$. This is obvious if $q \in H$ and follows from Lemma 3.4 if $q \in H(\infty)$. Now select points $q_1, \cdots, q_N \in \overline{H} - O^*$ such that $\overline{H} - O^* \subseteq \bigcup_{i=1}^N O_{q_i}$. If $O = \bigcap_{i=1}^N V_{q_i}$ and $R = \max\{R_{q_i} : 1 \leqslant i \leqslant N\}$, then O and R have the desired properties.

(3.6) Lemma. *Let $x \in H(\infty)$ be a hyperbolic point, and let $O^* \subseteq H(\infty)$ be a neighborhood of x. Then we can find a neighborhood $O \subseteq H(\infty)$ of x such that if γ is a geodesic of H with endpoints in O and $H(\infty) - O^*$, then γ is of rank one. Moreover each $y^* \in H(\infty) - O^*$ and $x^* \in O$ determine a rank one geodesic $\gamma_{x^*y^*}$.*

Proof. For each point $y \in H(\infty) - \{x\}$ there is a unique geodesic joining x to y and this geodesic is of rank one. For each $y \in H(\infty) - O^*$ there exist neighborhoods $O_y \subseteq H(\infty)$ of y and $V_y \subseteq H(\infty)$ of x such that if $y^* \in O_y$ and $x^* \in V_y$, then there is a unique geodesic joining x^* to y^* and this geodesic is of rank one (Lemma 3.4). By the compactness of $H(\infty) - O^*$ there exist points $y_1, \cdots, y_N$ in $H(\infty) - O^*$ such that $H(\infty) - O^* \subseteq \bigcup_{i=1}^N O_{y_i}$. Set $O = \bigcap_{i=1}^N V_{y_i}$.

(3.7) Lemma. *Let $x \in H(\infty)$ be a hyperbolic point, and let $O \subseteq \overline{H}$ be a neighborhood of x. Then*

$$\measuredangle_{p_n}(\overline{H} - O) = \sup\left\{\measuredangle_{p_n}(a, b) : a, b \in \overline{H} - O\right\} \to 0 \quad \text{as } n \to +\infty$$

for any sequence $\{p_n\} \subseteq H$ that converges to x.

Proof. Let $\{p_n\} \subseteq H$ converge to x and let $\{x_n\} \subseteq \overline{H} - O$ be an arbitrary sequence. It suffices to prove that $\measuredangle_{p_n}(p, x_n) \to 0$ as $n \to +\infty$, where $p \in H$ is a fixed point. This is clear if $\{x_n\} \subseteq H$ is bounded by the law of cosines (see for example [15, pp. 47–48]). We may assume that $x_n \to x^* \in \overline{H} - O$ by passing to a subsequence. Let σ, σ_n be the unique geodesics from x to x^* and from p_n to x_n respectively. By Lemma 3.4. $\sigma_n'(0) \to \sigma'(0)$ as $n \to +\infty$ for suitable parametrizations of σ_n. Hence $d(p, \sigma_n(0))$ is bounded and $\sphericalangle_{p_n}(p, x_n) = \sphericalangle_{p_n}(p, \sigma_n(0)) \to 0$ as $n \to \infty$ be the law of cosines.

(3.8) Corollary. *Let x, y be distinct points in $H(\infty)$, and let x be hyperbolic. Suppose that x, y are dual relative to some subgroup $\Gamma \subseteq I(H)$. Then for any neighborhoods $O_x \subseteq \overline{H}$ of x and $O_y \subseteq \overline{H}$ of y there exists an isometry $\varphi \in \Gamma$ such that*

$$\varphi\left(\overline{H} - O_x\right) \subseteq O_y \quad \textit{and} \quad \varphi^{-1}\left(\overline{H} - O_y\right) \subseteq O_x.$$

Proof. By hypothesis we can choose a sequence $\{\varphi_n\} \subseteq \Gamma$ so that $\varphi_n p \to y$ and $\varphi_n^{-1} p \to x$ as $n \to \infty$, where $p \in H$ is fixed. By Lemma 3.7

$$\sphericalangle_p\left(\varphi_n\left(\overline{H} - O_x\right)\right) = \sphericalangle_{\varphi_n^{-1}p}\left(\overline{H} - O_x\right) \to 0 \quad \text{as } n \to +\infty.$$

Since $\varphi_n p \to y$ and $d(p, \varphi_n(\overline{H} - O_x)) = d(\varphi_n^{-1}p, \overline{H} - O_x) \to \infty$ as $n \to \infty$ we may choose $\varphi = \varphi_n$ for any n sufficiently large.

(3.9) Lemma. *Let $x \in H(\infty)$ be a hyperbolic point, and let $O^* \subseteq \overline{H}$ be a neighborhood of x. There exists a neighborhood $O \subseteq \overline{H}$ of x such that if $\{q_n\} \subseteq H - O^*$ is any divergent sequence, then $\sphericalangle_{q_n}(O) \to 0$ as $n \to \infty$.*

Proof. Fix a point $p \in H$. By Lemma 3.5 there exists a number $R > 0$ and neighborhood $O \subseteq \overline{H}$ of x such that if σ is a geodesic in H with endpoints in O and $\overline{H} - O^*$, then $d(p, \sigma) \leqslant R$. It suffices to prove that $\sphericalangle_{q_n}(p, x_n) \to 0$ for any sequence $\{x_n\} \subseteq 0$ and any divergent sequence $\{q_n\} \subseteq H - O^*$. If σ_n denotes the geodesic from q_n to x_n, then we may choose q_n^* on σ_n so that $d(p, q_n^*) \leqslant R$ for every n. Then $\sphericalangle_{q_n}(p, x_n) = \sphericalangle_{q_n}(p, q_n^*) \to 0$ as $n \to \infty$ by the law of cosines.

(3.10) Lemma. *Let H have rank one, and let A, B be open sets in $H(\infty)$ such that $\sphericalangle_p(a, b) \geqslant \delta > 0$ for all $a \in A$, $b \in B$, some point $p \in H$ and some $\delta > 0$. Let x_1, x_2 be distinct hyperbolic points in $H(\infty)$. Then we can find disjoint neighborhoods V_1, V_2 in $H(\infty)$ of x_1, x_2 such that for each $\varphi \in I(H)$ at least one of the following holds*:

(1) $\varphi(V_1) \cap A = \varnothing$.
(2) $\varphi(V_1) \cap B = \varnothing$.
(3) $\varphi(V_2) \cap A = \varnothing$.
(4) $\varphi(V_2) \cap B = \varnothing$.

Proof. Fix disjoint neighborhoods W_1, W_2 of x_1, x_2 in $\overline{H}$. Choose neighborhoods O_1, O_2 of x_1, x_2 in $\overline{H}$ with $\overline{O}_i \subseteq W_i$ for $i = 1, 2$. By Lemma 3.9 we can choose neighborhoods U_1, U_2 of x_1, x_2 in $H(\infty)$ such that $U_i \subseteq O_i$ and $\sphericalangle_{p_n}(p, U_i) \to 0$ as $n \to \infty$ for any divergent sequence $\{p_n\} \subseteq H - O_i$ for $i = 1, 2$. Suppose now that the assertion is false. Then for $i = 1, 2$ we can find nested neighborhood bases $\{V_n^i\}$ at x_i in $H(\infty)$ such that $V_n^i \subseteq U_i$ for all n, and a sequence $\{\varphi_n\} \subseteq I(H)$ such that $\varphi_n(V_n^i) \cap A \neq \varnothing$ and $\varphi_n(V_n^i) \cap B \neq \varnothing$ for all n.

We show first that $\varphi_n^{-1}(p)$ has a cluster point in $H(\infty)$. If this were not the case, then $\{\varphi_n\}$ would converge to some element $\varphi \in I(H)$, passing to a subsequence if necessary [21, p. 167]. The sets $\varphi_n(V_n^i)$ would then ultimately lie in any given neighborhood in $H(\infty)$ of $\varphi(x_i)$ since $\{V_n^i\}$ is a neighborhood basis at x_i. In particular $\varphi_n(V_n^i)$ could not intersect the δ-separated sets A, B for large n, which contradicts the choice of φ_n.

Now let $z \in H(\infty)$ be a cluster point of $\{\varphi_n^{-1}(p)\}$. Either $z \in H(\infty) - W_1$ or $z \in H(\infty) - W_2$ since W_1, W_2 are disjoint. Assume for simplicity that $z \in H(\infty) - W_1$ and choose a subsequence $\{\varphi_{n_k}\}$ so that $\varphi_{n_k}^{-1}(p) \to z$ as $k \to +\infty$. Assume further that $\varphi_{n_k}(p)$ converges to a point $z^* \in H(\infty)$. Then $\{\varphi_{n_k}^{-1}(p)\} \subseteq H - O_1$ for large k by the definition of O_1 and hence $\measuredangle_p(\varphi_{n_k}(p), \varphi_{n_k}(U_1)) = \measuredangle_{\varphi_{n_k}^{-1}(p)}(p, U_1) \to 0$ as $k \to +\infty$ by the choice of U_1. Hence the sets $\varphi_{n_k}(U_1)$ converge to z^*, contradicting the fact that $\varphi_{n_k}(U_1)$ contains $\varphi_{n_k}(V_{n_k}^1)$ and hence meets both A and B for all k.

Proof of Theorem 3.1. Let x, y, z, w be distinct points in $H(\infty)$ such that z, w are hyperbolic. Given $p \in H$ fix $\delta > 0$ and open neighborhoods A, B, C, D of x, y, z, w in $H(\infty)$ such that $\measuredangle_p(q_1, q_2) \geqslant \delta > 0$ for any points q_1, q_2 taken from two different sets of the four possibilities. We assume furthermore that C, D satisfy the conclusions of Lemma 3.10 relative to A and B. Now choose neighborhoods V_1, V_2 in $H(\infty)$ of z, w so that $\overline{V}_1 \subseteq C$, $\overline{V}_2 \subseteq D$ and for any points z_1, z_2 in V_1, $H(\infty) - C$ (respectively in V_2, $H(\infty) - D$) there exists a unique rank one geodesic of H with endpoints z_1, z_2. This can be done by Lemma 3.6. Since Γ satisfies the duality condition and H has rank one, any two points of $H(\infty)$ are dual relative to Γ; the proof is implicit in [2] and follows explicitly from Lemma 2.3b of [9] and the discussion of (1.5).

Choose elements φ_1, ψ_1, φ_2, ψ_2 in Γ such that

(a) $\varphi_1(H(\infty) - A) \subseteq V_1$ and $\varphi_1^{-1}(H(\infty) - V_1) \subseteq A$,
(b) $\psi_1(H(\infty) - A) \subseteq V_2$ and $\psi_1^{-1}(H(\infty) - V_2) \subseteq A$,
(c) $\varphi_2(H(\infty) - B) \subseteq V_1$ and $\varphi_2^{-1}(H(\infty) - V_1) \subseteq B$,
(d) $\psi_2(H(\infty) - B) \subseteq V_2$ and $\psi_2^{-1}(H(\infty) - V_2) \subseteq B$.

This can be done by Corollary 3.8. Now let $\xi \in \Gamma$ be given arbitrarily. One of the four possibilities of Lemma 3.10 must occur for ξ relative to A, B, C, D by the definition of these sets. For simplicity we consider first the case that $\xi(C) \cap A = \varnothing$. Then

$$(*) \qquad (\varphi_1\xi)(C) \subseteq V_1 \subseteq \overline{V}_1 \subseteq C$$

and hence $\varphi_1\xi$ has a fixed point z_1 in V_1. (We may assume that all neighborhoods considered are homeomorphic to open disks in $\mathbb{R}^{n-1}$.) Observe that

$$\begin{aligned}(\varphi_1\xi)^{-1}(H(\infty) - C) &= (\xi^{-1}\varphi_1^{-1})(H(\infty) - C) \\ &\subseteq (\xi^{-1}\varphi_1^{-1})(H(\infty) - V_1) \subseteq \xi^{-1}(A) \subseteq H(\infty) - C\end{aligned}$$

and hence $\varphi_1\xi$ has a fixed point z_2 in $H(\infty) - C$. Therefore $\varphi_1\xi$ leaves invariant the unique rank one geodesic γ^* joining z_1 to z_2. Moreover, $\varphi_1\xi$ fixes no point p of γ^*; if $\varphi_1\xi$ did fix a point p of γ^*, then by identifying C, V_1, $\overline{V}_1$ with subsets of the unit sphere in T_pH the inclusion relations of $(*)$ would imply that the differential map of $\varphi_1\xi$ at p strictly compresses the set C, which is impossible for an element of $O(n)$, $n = \dim H$. Therefore $\varphi_1\xi$ translates γ^* and $\varphi_1\xi \in B_1 \subseteq A_1$.

We have shown that if $\xi(C) \cap A = \varnothing$ for an element $\xi \in \Gamma$, then $\varphi_1\xi \in B_1 \subseteq A_1$. The other three possibilities from Lemma 3.10 are that $\xi(C) \cap B = \varnothing$, $\xi(D) \cap A = \varnothing$, or $\xi(D) \cap B = \varnothing$. Arguing as above we show respectively that $\varphi_2\xi$, $\psi_1\xi$, or $\psi_2\xi$ are elements of B_1. It follows that

$$\Gamma = \varphi_1^{-1}A_1 \cup \varphi_2^{-1}A_1 \cup \psi_1^{-1}A_1 \cup \psi_2^{-1}A_1$$

which implies that $r(\Gamma) \leqslant 1$. (If Γ has no elements of finite order, then $r(\Gamma) = 1$ since then A_0 is empty.)

We complete the proof by showing that $r(\Gamma) = 1$. Suppose that this is false and write

$$(+) \qquad \Gamma = \xi_1 A_0 \cup \xi_2 A_0 \cup \cdots \cup \xi_k A_0$$

for suitable elements $\xi_1, \cdots, \xi_k \in \Gamma$. Note that A_0 contains only elements of finite order and each of these elements must be elliptic (have a fixed point in H). (However, not all elliptic elements of Γ may belong to A_0.) Now choose distinct points x, y, z, w in $H(\infty)$ with z, w hyperbolic and choose neighborhoods A, B, C, D in $H(\infty)$ of these points as above so that $\xi_i(C) \cap A = \varnothing$ for all i, $1 \leqslant i \leqslant k$. This is possible since there are only finitely many of the ξ_i and since hyperbolic points are dense in $H(\infty)$. Choose an element $\varphi_1 \in \Gamma$ satisfying (a) above. The arguments above show that $\varphi_1\xi_i$ translates a geodesic for each i, $1 \leqslant i \leqslant k$, and in particular $\varphi_1\xi_i$ is not elliptic. On the other hand, by $(+)$ there is an i such that $\varphi_1^{-1} \in \xi_i A_0$. This implies that $\xi_i^{-1}\varphi_1^{-1} = (\varphi_1\xi_i)^{-1} \in A_0$ and hence both $\varphi_1\xi_i$ and its inverse are elliptic, a contradiction. Therefore $r(\Gamma) = 1$.

The discussion above completes the treatment of the rank one case of Theorem B as stated in the introduction. We now complete the proof of that result and actually generalize it slightly.

(3.11) Theorem. *Let H be a complete simply connected Riemannian manifold with sectional curvature satisfying $-a^2 \leqslant K \leqslant 0$ for some positive constant a. If Γ is a discrete group of isometries of H such that* $\mathrm{vol}(H/\Gamma) < \infty$, *then* $\mathrm{rank}(\Gamma) = \mathrm{rank}(H)$.

Proof. If H is a symmetric space of noncompact type, then a finite index subgroup Γ^* of Γ is contained in $I_0(H)$, and we conclude from the result of Prasad and Raghunathan [27, Theorem 3.9] that

$$\mathrm{rank}(\Gamma) = \mathrm{rank}(\Gamma^*) = r(\Gamma^*) = \mathrm{rank}(H).$$

Next suppose that H admits no Euclidean de Rham factor and is not a symmetric space of noncompact type. By Theorem 1.8 we may write H as a Riemannian product $H_S \times H_1 \times \cdots \times H_k$, where H_S is a symmetric space of noncompact type and each H_i for $1 \leqslant i \leqslant k$ is a rank one manifold whose isometry group is discrete and satisfies the duality condition. Of course, the factor H_S may be absent. By Theorem 1.7 there exists a finite index subgroup Γ^* of Γ such that Γ^* is a direct product $\Gamma_S \times \Gamma_1 \times \cdots \times \Gamma_k$, where Γ_S, $\Gamma_1, \cdots, \Gamma_k$ are discrete subgroups of $I(H_S)$, $I(H_1), \cdots, I(H_k)$ which satisfy the duality condition. It follows that $\mathrm{vol}(H_S/\Gamma_S) < \infty$ and $\mathrm{vol}(H_i/(\Gamma_i) < \infty$ for $1 \leqslant i \leqslant k$ since $\mathrm{vol}(H/\Gamma^*) < \infty$. By Proposition 2.1, Theorem 3.1, and the symmetric space case just considered we conclude that

$$\begin{aligned}\mathrm{rank}(\Gamma) = \mathrm{rank}(\Gamma^*) &= \mathrm{rank}(\Gamma_S) + \sum_{i=1}^{k} \mathrm{rank}(\Gamma_i) \\ &= \mathrm{rank}(H_S) + k = \mathrm{rank}(H).\end{aligned}$$

Suppose now that H has a nontrivial Euclidean de Rham factor of dimension $k \geqslant 1$ and write $H = E^k \times H_1$ where E^k denotes k-dimensional Euclidean space and H_1 denotes the product of all non-Euclidean de Rham factors of H. By Lemma 3 of [11] we may choose a finite index subgroup Γ_0 of Γ such that if Γ^* is any finite index subgroup of Γ_0, then $Z(\Gamma^*)$, the center of Γ^*, equals $C(\Gamma^*)$, the Clifford subgroup of Γ^* consisting of all elements of the form $T \times \{1\}$ in $I(H) = I(E^k) \times I(H_1)$, where T is a translation of E^k. Moreover, it follows from the main theorem of [13] that $Z(\Gamma^*)$ is a free abelian group of rank k and hence that $E^k/Z(\Gamma^*)$ is a flat k-torus. We remark that the results just quoted from [11], [13] are actually stated there under slightly stronger hypotheses on Γ. However, the results remain true under our hypotheses on Γ, and the proofs require no modification.

Now let Γ^* be any finite index subgroup of Γ_0, and let $p : \Gamma^* \to I(E^k)$ and $p_1 : \Gamma^* \to I(H_1)$ denote the projection homomorphisms. The group $p(\Gamma^*)$ consists of translations of E^k since $Z(\Gamma^*)$ is a lattice of translations in E^k.

The group $\Gamma_1 = p_1(\Gamma^*)$ is a discrete subgroup of $I(H_1)$ by Lemma A in §2 of [13]. The projection of H onto H_1 induces a projection P of the quotient spaces H/Γ^* onto H_1/Γ_1, where both spaces are smooth Riemannian manifolds except on a subset of measure zero. The fibers of P are isometric, totally geodesic smooth immersions of the flat k-torus $E^k/Z(\Gamma^*)$ since $Z(\Gamma^*) = C(\Gamma^*)$ is the kernel of p_1. Hence $\mathrm{vol}(H_1/\Gamma_1) < \infty$ since $\mathrm{vol}(H/\Gamma^*) < \infty$ and by the first part of the proof we get

$$\mathrm{rank}(\Gamma_1) = \mathrm{rank}(H_1) = \mathrm{rank}\,(H) - k.$$

Now consider an element $\varphi = \varphi_e \times \varphi_1$ in Γ^*, where $\varphi_e \in p(\Gamma^*)$ and $\varphi_1 \in \Gamma_1 = p_1(\Gamma^*)$. Since $p(\Gamma^*)$ is a group of translations of E^k we see that $p_1^{-1}(Z_{\Gamma_1}(\varphi_1)) = Z_{\Gamma^*}(\varphi)$ and $p_1 \colon Z_{\Gamma^*}(\varphi) \to Z_{\Gamma_1}(\varphi_1)$ is a surjective homomorphism with kernel $Z(\Gamma^*)$. Moreover, $p_1^{-1}(\Delta) \subseteq \Gamma^*$ is an abelian subgroup if and only if $\Delta \subseteq \Gamma_1$ is an abelian subgroup. It follows that $p_1(A_i(\Gamma^*)) = A_{i-k}(\Gamma_1)$ and $A_i(\Gamma^*) = p_1^{-1}(A_{i-k}(\Gamma_1))$, which implies that $A_i(\Gamma^*)$ is invariant under multiplication by elements of kernel $(p_1) = Z(\Gamma^*)$. Therefore

$$\Gamma^* = \bigcup_{\alpha=1}^{N} \varphi_\alpha A_i(\Gamma^*) \quad \text{for a finite subset } \{\varphi_\alpha\} \subseteq \Gamma^*$$

if and only if

$$\Gamma_1 = p_1(\Gamma^*) = \bigcup_{\alpha=1}^{N} p_1(\varphi_\alpha) A_{i-k}(\Gamma_1).$$

We conclude that $\mathrm{rank}(\Gamma) = \mathrm{rank}(\Gamma^*) = \mathrm{rank}(\Gamma_1) + k = \mathrm{rank}(H)$.

We conclude this section with the proof of Theorem C as stated in the Introduction. Suppose that M is an irreducible locally symmetric space of noncompact type and rank $k \geqslant 2$. The first condition is well known (see for example Theorem 13.14 of [28]). Conditions (2) and (3) are necessary by the splitting result Theorem 1.6 and the results of Prasad and Raghunathan respectively. Suppose next that M is a space of finite volume and bounded sectional curvature that satisfies the three conditions given. The first two conditions and Theorem 1.9 imply that the universal cover H of M admits no Euclidean de Rham factor. The second condition implies that M is irreducible. Theorems 1.7, 1.8, and the second condition imply that either H is a rank one space or H is a symmetric space of noncompact type. The third condition and Theorem 3.1 imply that $\mathrm{rank}(H) = k \geqslant 2$.

4. Some examples

We first discuss two examples that show that $r(\Gamma)$ may be strictly smaller than $\mathrm{rank}(\Gamma)$, even if H/Γ is locally symmetric.

Example 1. Let Γ be the fundamental group of a flat Klein bottle, acting by isometries on the Euclidean plane E^2. (In the simplest case Γ is generated by $\varphi_1:(x, y) \to (x+1, -y)$ and $\varphi_2:(x, y) \to (x, y+1)$.) The set A_1 consists of all elements of Γ that reverse the orientation of E^2. Since the map $\Gamma \to \mathbf{Z}_2$ sending orientation preserving elements to 0 and orientation reversing elements to 1 is a homomorphism and since A_1 is the preimage of 1, it follows that $\Gamma = eA_1 \cup \gamma A_1$, where $\gamma \in A_1$ is arbitrary. Therefore $r(\Gamma) < 2 = \mathrm{rank}(\Gamma) = \mathrm{rank}(E^2)$. Similarly it can be shown that a Bieberbach group Γ of rank k, that is, a discrete uniform group Γ of isometries of k-dimensional Euclidean space E^k, satisfies

$$r(\Gamma) < k = \mathrm{rank}(\Gamma) = \mathrm{rank}(E^k)$$

unless Γ is free abelian of rank k. Note, however, that a Bieberbach group of rank k always has a free abelian group of rank k as a subgroup of finite index.

Example 2. Let H denote the hyperbolic plane with Gaussian curvature -1. If we regard H as the upper half-plane with metric $ds^2 = (1/y^2)(dx^2 + dy^2)$, then $I_0(H)$ is the group $\mathrm{PSL}(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R})/\{\pm I\}$ acting by fractional linear transformations. We now construct a lattice subgroup Γ_0 of $I(H \times H)$ with $r(\Gamma_0) = 1$ and $\mathrm{rank}(\Gamma_0) = \mathrm{rank}(H \times H) = 2$ such that $H \times H/\Gamma_0$ is a smooth, nonorientable, locally symmetric manifold of finite volume, which may be chosen to be either compact or noncompact. This example does not contradict the result of Prasad and Raghunathan mentioned earlier, since $\Gamma_0 \not\subset I_0(H \times H) = \mathrm{PSL}(2, \mathbf{R}) \times \mathrm{PSL}(2, \mathbf{R})$.

Let $\Gamma \subseteq \mathrm{PSL}(2, \mathbf{R})$ be a lattice, either uniform or nonuniform, with no elements of finite order. For any element $\tau \neq 1$ in Γ there exists a normal subgroup Γ^* of finite index such that $\tau \in \Gamma - \Gamma^*$ (residual finiteness of Γ). Choose Γ, Γ^*, and τ so that the image of τ in Γ/Γ^* has even order. The isometry group of $(H/\Gamma^*) \times (H/\Gamma^*)$ is finite and isomorphic to $N(\Gamma^* \times \Gamma^*)/\Gamma^* \times \Gamma^*$, where $N(\Gamma^* \times \Gamma^*)$ denotes the normalizer in $I(H \times H)$ of $\Gamma^* \times \Gamma^*$. Let $\mu \in N(\Gamma^* \times \Gamma^*)$ be the element given by $\mu(x, y) = (\tau y, x)$, and let Γ_0 be the finite index subgroup of $N(\Gamma^* \times \Gamma^*)$ generated by $\Gamma^* \times \Gamma^*$ and μ.

We assert that no nonidentity element of Γ_0 has a fixed point in $H \times H$ and that the coset $(\Gamma^* \times \Gamma^*) \cdot \mu$ lies entirely in $A_1(\Gamma_0)$. This will show that $H \times H/\Gamma_0$ is a smooth manifold with $r(\Gamma_0) = 1$ although $\mathrm{rank}(\Gamma_0) = \mathrm{rank}(\Gamma^* \times \Gamma^*) = 2$.

We show first that no element of Γ_0 has a fixed point in $H \times H$. Since μ normalizes $\Gamma^* \times \Gamma^*$, every element of Γ_0 has the form $(\varphi_1 \times \varphi_2) \cdot \mu^r$, where $\varphi_1 \times \varphi_2 \in \Gamma^* \times \Gamma^*$ and r is an integer. If $(\varphi_1 \times \varphi_2) \cdot \mu^{2k} = (\varphi_1 \times \varphi_2) \cdot (\tau^k \times \tau^k)$ fixes a point (x, y), then $\varphi_1 \tau^k x = x$ and $\varphi_2 \tau^k y = y$. Since Γ has no elliptic elements it follows that $\varphi_1 \tau^k = \varphi_2 \tau^k = 1$ and hence $(\varphi_1 \times \varphi_2) \cdot \mu^{2k} = 1$. An element of the form $(\varphi_1 \times \varphi_2) \cdot \mu^{2k+1}$ fixes a point (x, y) if and only if $\varphi_1 \tau^{k+1} y = x$ and $\varphi_2 \tau^k x = y$. In this case $(\varphi_2 \tau^k \varphi_1 \tau^{-k}) \tau^{2k+1} = \varphi_2 \tau^k \varphi_1 \tau^{k+1}$ fixes y and hence must be the identity since Γ contains no elliptic elements. Therefore $\tau^{2k+1} \in \Gamma^*$ since τ normalizes Γ^*, but this contradicts the assumption that the image of τ in Γ/Γ^* has even order.

We next consider the centralizer in Γ_0 of an element in $(\Gamma^* \times \Gamma^*) \cdot \mu$. Let $\varphi = (\varphi_1 \times \varphi_2) \cdot \mu$ be such an element and let $G = Z_{\Gamma_0}(\varphi) \cap (\Gamma^* \times \Gamma^*)$, a subgroup of finite index in $Z_{\Gamma_0}(\varphi)$. To show that $(\Gamma^* \times \Gamma^*) \cdot \mu \subseteq A_1(\Gamma_0)$ it suffices to show that G is infinite cyclic. An element $\xi_1 \times \xi_2 \in \Gamma^* \times \Gamma^*$ commutes with $(\varphi_1 \times \varphi_2) \cdot \mu \in (\Gamma^* \times \Gamma^*) \cdot \mu$ if and only if

(a) $\xi_1 \varphi_1 \tau = \varphi_1 \tau \xi_2$ and

(b) $\xi_2 \varphi_2 = \varphi_2 \xi_1$.

These conditions imply that ξ_1 commutes with $\varphi_1 \tau \varphi_2$, which is a nonidentity element of Γ since $\tau \in \Gamma - \Gamma^*$. Note that $Z_\Gamma(\varphi_1 \tau \varphi_2)$ is infinite cyclic since Γ is a discrete subgroup of PSL(2, **R**) with no elements of finite order. If $\alpha \in \Gamma$ is a generator for $Z_\Gamma(\varphi_1 \tau \varphi_2)$, then $\xi_1 = \alpha^m$ for some integer m and $\xi_2 = \varphi_2 \xi_1 \varphi_2^{-1} = (\varphi_2 \alpha \varphi_2^{-1})^m$ by (b) above. Therefore G is a subgroup of the infinite cyclic group with generator $\alpha \times (\varphi_2 \alpha \varphi_2^{-1})$.

Remark. If the image of τ in Γ/Γ^* has odd order, then the group Γ_0 constructed above always has elliptic elements.

Example 3. This is an example where Γ satisfies (2) in Theorem C of the Introduction and rank$(\Gamma) = 2$, but where M is not locally symmetric, showing that condition (1) cannot be deleted. This example also shows that the hypothesis in Theorem 1.9 that Γ be finitely generated cannot be deleted. Our construction is similar to one given by Gromov [17].

For all $m \geqslant 2$, let F_m be a surface of nonpositive curvature bounded from below by some constant $-a^2$ and finite volume $\leqslant m \cdot C$ such that

(i) a neighborhood of the boundary of F_m is isometric to a disjoint union of m copies of $[0, \varepsilon_m) \times S^1_{r_{m-1}}$ and m copies of $[0, \varepsilon_m) \times S^1_{r_{m+1}}$, where $r_m = 2^{-m}$ and S^1_r denotes a circle of radius r.

(ii) $\mathbf{Z}_m$ operates isometrically on F_m with a fixed point p_m (there may be others as well) such that $F_m/\mathbf{Z}_m$ has two boundary components, a neighborhood of one component being isometric to $[0, \varepsilon_m) \times S^1_{r_{m-1}}$ and a neighborhood of the other one being isometric to $[0, \varepsilon_m) \times S^1_{r_{m+1}}$.

For $m \geqslant 2$ let

$$V_m = \left(F_m \times S_1^1 \times S_{r_m}^1\right)/\mathbf{Z}_m,$$

where $\mathbf{Z}_m$ acts by $\gamma(p,t,s) = (\gamma(p),\gamma(t),s)$ and $\gamma(t)$ denotes the canonical action of Z_m on S_1^1. Note that V_m, $m \geqslant 2$, has two boundary components, a neighborhood of one component isometric to

$$[0,\varepsilon_m) \times S_{r_{m-1}}^1 \times S_1^1 \times S_{r_m}^1$$

and a neighborhood of the other component isometric to

$$[0,\varepsilon_m) \times S_{r_{m+1}}^1 \times S_1^1 \times S_{r_m}^1.$$

Hence V_m and V_{m+1} can be glued together along the appropriate boundary components by the canonical isometry of

$$S_{r_{m+1}}^1 \times S_1^1 \times S_{r_m}^1 \quad \text{with } S_{r_m}^1 \times S_1^1 \times S_{r_{m+1}}^1.$$

This glueing procedure, performed for all $m \geqslant 2$, yields a manifold V_0 with one boundary component, a neighborhood of which is isometric to

$$[0,\varepsilon_2) \times S_{r_1}^1 \times S_1^1 \times S_{r_2}^1.$$

Take an isometric copy V_1 of V_0 and glue V_0 and V_1 along their boundaries. As a result we obtain a complete Riemannian manifold M without boundary and of nonpositive curvature bounded from below by $-a^2$. Note that the volume of M is finite since

$$\operatorname{vol}(V_m) \leqslant C \cdot 4\pi^2 \cdot 2^{-m}.$$

The rank of Γ, the fundamental group of M, is two. H contains a Euclidean factor of dimension one, coming from the S_1^1's.

We conclude the discussion of the example by showing that Γ has no finite index subgroup Γ^* that is a nontrivial direct product $\Gamma_1 \times \Gamma_2$, and in particular this shows that no finite cover of $M = H/\Gamma$ can be diffeomorphic to $S^1 \times M'$, where M' has dimension 3. We argue by contradiction and assume that Γ does admit such a finite index subgroup $\Gamma^* = \Gamma_1 \times \Gamma_2$. The group Γ^* satisfies the duality condition by (1.5), and it follows from the corollary to theorem 1 of [29] that H admits a Γ^*-invariant splitting

$$H = H_0 \times H_1 \times H_2$$

which satisfies the following conditions: (1) H_0 is a Euclidean space of dimension $s \geqslant 0$. (2) Every element ϕ of Γ_1 can be written as $\phi = (\phi_0, \phi_1, 1) \in I(H_0) \times I(H_1) \times I(H_2)$ and every element ψ of Γ_2 can be written as $\psi = (\psi_0, 1, \psi_2)$, where ϕ_0 and ψ_0 are translations on H_0. Of course, any one of the factors H_0, H_1, H_2 may be trivial but at least two of the factors must be nontrivial. We consider separately the cases (1) one of the factors H_1, H_2 is trivial, and (2) the factors H_1, H_2 are both nontrivial. In the discussion below let $p_i : \Gamma \to I(H_i)$ denote the projection homomorphism for $0 \leqslant i \leqslant 2$.

In case (1) we may assume without loss of generality that the factor H_2 is trivial. Hence $H = H_0 \times H_1$, where H_0 has dimension 1 and H_1 has no Euclidean factor. Moreover $\phi = (\varphi_0, \phi_1)$ for $\phi \in \Gamma_1$ and $\psi = (\psi_0, 1)$ for $\psi \in \Gamma_2$, where ϕ_0 and ψ_0 are translations on H_0. We show first that $\Gamma_2 = Z^*$, the center of Γ^*. Clearly $\Gamma_2 \subset Z^*$ since translations on H_0 commute. To prove the reverse inclusion observe that $p_1(\Gamma^*) \subset I(H_1)$ satisfies the duality condition in H_1 by (1.5) and hence $p_1(\Gamma^*)$ has trivial center by Theorem 4.2 of [8] and the fact that H_1 has no Euclidean de Rham factor. It follows that an element $\phi \in Z^*$ has the form $\phi = (\phi_0, 1)$, where ϕ_0 is a translation on H_0. Hence Z^* is an infinite cyclic group and the subgroup Γ_2 is also infinite cyclic with finite index k for some integer $k \geqslant 1$. If $\phi = (\phi_0, 1)$ is an arbitrary element of Z^*, then we write $\phi = \psi_1 \psi_2$, where $\psi_i \in \Gamma_i$ for $i = 1, 2$. Since $\psi_1 = \phi \psi_2^{-1} \in Z^*$ it follows that $\psi_1^k \in \Gamma_2 \cap \Gamma_1 = \{1\}$ and hence $\psi_1 = 1$ since every element of Γ^* has infinite order. Therefore $Z^* \subset \Gamma_2$ and equality follows.

For each positive integer m there exists a point $q_m \in M = H/\Gamma$ and a simple, closed geodesic $\gamma_m \subset M$ of length $2\pi/m$ that passes through q_m and is tangent to the 1-dimensional local Euclidean de Rham factor of M. To construct q_m and γ_m let $q_m^* = (p_m \alpha, \beta) \in V_m^* = F_m \times S_1^1 \times S_{r_m}^1$, where α, β are arbitrary and $p_m \in F_m$ is a point fixed by $\mathbf{Z}_m$. Let $\gamma_m^* \subset V_m^*$ be the circle of length 2π through q_m^* that is tangent to the S_1^1 factor of V_m^*. Now let q_m, γ_m be the projections of q_m^*, γ_m^* into $V_m = V_m^*/\mathbf{Z}_m$.

Let $\tilde{\gamma}_m \subset H$ be a lift of γ_m, and let $\phi_m \in \Gamma$ be an element that translates $\tilde{\gamma}_m$ by an amount $\pm 2\pi/m$. Choose an integer N_m such that $1 \leqslant N_m \leqslant [\Gamma : \Gamma^*]$ and $\phi_m^{N_m} \in \Gamma^*$. It follows from the construction of γ_m that $\tilde{\gamma}_m(t) = (t + t_m, r_m) \in H_0 \times H_1$ for all $t \in \mathbf{R}$, some $t_m \in H_0 = \mathbf{R}$ and some $r_m \in H_1$. Since $\phi_m^{N_m}$ translates $\tilde{\gamma}_m$ by $\pm 2\pi N_m/m$ it follows from property (2) of the splitting $H = H_0 \times H_1 \times H_2$ that we may write

$$\phi_m^{N_m} = (\alpha_m, \beta_m) \in I(H_0) \times I(H_1),$$

where α_m is a translation on H_0 by $\pm 2\pi N_m/m$ and β_m fixes r_m. The group $p_1(\Gamma) \subset I(H_1)$ is discrete by Lemma A of [13], and hence $\beta_m^{k_m} = 1$ for some positive integer k_m. Now write

$$\phi_m^{N_m} = \phi_m^* \phi_m^{**},$$

where $\phi_m^* \in \Gamma_1$ and $\phi_m^{**} \in \Gamma_2$. It follows that $\phi_m^{**} = (\alpha_m^{**}, 1)$ for some translation α_m^{**} on H_0 and $\phi_m^* = (\alpha_m(\alpha_m^{**})^{-1}, \beta_m)$. In particular $(\phi_m^*)^{k_m} \in Z^* = \Gamma_2$ and hence $\phi_m^* = 1$ since $\Gamma_1 \cap \Gamma_2 = \{1\}$ and all elements of Γ^* have infinite order. Hence $\phi_m^{N_m} = (\alpha_m, 1) \in Z^* = \Gamma_2$ for every m. However, if $\phi^* = (\alpha, 1)$ is a generator for Z^*, where α is a translation by $a > 0$ on H_0, then $\phi_m^{N_m} = (\phi^*)^{j_m}$ for some nonzero integer j_m and hence $\alpha_m = \alpha^{j_m}$ is a translation on H_0 by the amount $j_m a$ for every m. This contradicts the fact that α_m is a translation on H_0 by $\pm 2\pi N_m/m \to 0$ as $m \to \infty$ since $1 \leqslant N_m \leqslant [\Gamma : \Gamma^*]$. Hence case (1) cannot arise.

Next, we consider case (2) in which both factors H_1 and H_2 are nontrivial. The fact that H has dimension 4 and a Euclidean factor of dimension 1 implies that the factor H_0 is trivial, and hence $M^* = H/\Gamma^*$ is isometric to the Riemannian product $(H_1/\Gamma_1) \times (H_2/\Gamma_2)$ by property (2) of the splitting $H = H_0 \times H_1 \times H_2$. One of these factors H_1 or H_2, say the former, must have a Euclidean factor since H does. H would have a 2-dimensional Euclidean factor if H_1 had either dimension 2, in which case H_1 is Euclidean, or dimension 3, in which case H_2 has dimension 1. Therefore H_1 has dimension 1 and through all points of M^* there are simple parallel closed geodesics of constant length $a > 0$ that are tangent to the local Euclidean factor of M^*. However, the argument above in the discussion of case (1) shows that for every positive integer m there exists a positive integer N_m with $1 \leqslant N_m \leqslant [\Gamma : \Gamma^*]$ and a closed geodesic $\gamma_m^* \subset M^*$ of length $2\pi N_m/m$ which is tangent to the local Euclidean factor of M^*. In particular, if m is sufficiently large, then $2\pi N_m/m < a$ and it is impossible to have a simple closed geodesic in M^* of length a that is tangent to the local Euclidean factor of M^* and passes through a point of γ_m^*; such a closed geodesic would be tangent to and hence a reparametrization of γ_m^*. This contradiction completes the proof that Γ admits no finite index subgroup $\Gamma^* = \Gamma_1 \times \Gamma_2$.

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